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ENERGY RELEASE RATE CALCULATIONS FOR INTERFACE EDGE CRACKS BASE--ETC(U)

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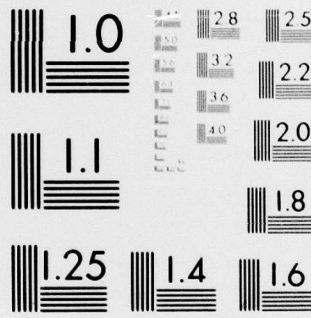
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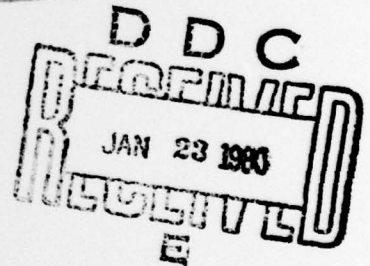
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14 MRC-TSR-2017  
Mathematics Research Center  
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November 1979 12/14

Received September 14, 1979

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DAAG25-75-C-0424  
F49620-79-C-0076

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ENERGY RELEASE RATE CALCULATIONS FOR INTERFACE EDGE CRACKS  
BASED ON A CONSERVATION INTEGRAL

A. Nachman<sup>†</sup> and J. R. Walton<sup>‡</sup>

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ABSTRACT

A certain conservation integral, the so called M-integral, recently exploited by L. B. Freund in the calculation of stress intensity factors for cracks in homogeneous elastic media, is applied to the calculation of energy release rates for interface edge cracks. Specifically, for an edge crack along the interface between two elastic wedges of different opening angles and dissimilar elastic properties, and that is subjected to point loads at the apex, a relation is derived among the length of the crack, the energy release rate of the crack, the applied loads, the wedge angles and the material parameters.

AMS(MOS) Subject Classifications - 73M05, 73K20

Key Words: Energy release rate, interface edge crack,  
M-conservation integral

Work Unit No. 3 - Applications of Mathematics

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Sponsored by the United States Army under Contract No. DAAG29-75-C-0024. Supported in part by the United States Air Force under AFOSR Contract F49620-79-C-0076 and under AFOSR Grant 77-3290.

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## SIGNIFICANCE AND EXPLANATION

The study of the strength of composite materials containing cracks or flaws is of great importance in structural mechanics. Of particular interest are elastic interface cracks, that is, cracks that have formed along the interface between two bonded dissimilar elastic bodies. An important parameter for the prediction of crack advance and ultimate bond failure is the energy release rate of the crack, that is, the rate of change of the energy of the body per unit advance of the crack. This report illustrates how a certain conservation integral may be used to calculate the energy release rate for the 2-dimensional problem of a crack along the interface between two bonded dissimilar elastic wedges, thereby obviating a consideration of the very complicated associated mixed boundary value problem.

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ENERGY RELEASE RATE CALCULATIONS FOR INTERFACE EDGE CRACKS  
BASED ON A CONSERVATION INTEGRAL

A. Nachman<sup>†</sup> and J. R. Walton<sup>‡</sup>

Introduction

In a recent paper, L. B. Freund [1] demonstrated the usefulness of the M-integral conservation law in the determination of stress intensity factors for 2-dimensional cracks in homogeneous elastic bodies with certain geometric properties. For a detailed discussion of the M-integral and its special features that permit the applications given by Freund, the reader is referred to that paper and the references contained therein. Liberal use will be made in this paper of many of the results derived by Freund.

What is demonstrated here, is that the M-integral can be used to determine the energy release rate for certain interface cracks in much the same way as for cracks in homogeneous bodies.

Three observations are needed for this application. The first, due to Smelzer and Gurtin [2], is that the J-integral on a small arc about an interface crack is equal to the energy release rate, as in the homogeneous case. One difference for cracks in homogeneous bodies is that the J-integral may also be related to the stress intensity factor, while for interface cracks, the J-integral is related to a composite stress intensity factor which has dubious utility. (See Smelzer and Gurtin [2].) The second observation involves the nature of the far stress field for bonded dissimilar elastic wedges, and appeals to the analysis presented by Bogy [3] for bonded dissimilar elastic quarter planes. The third observation is that the integrand of the M-integral is continuous across bonded interfaces lying along radial lines of the chosen coordinate system.

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## Statement and Analysis of the Problem

We present here an extension of the result obtained by Freund for the elastic half-space with an edge crack whose corners are subjected to normal and shear point loads. Specifically, the M-integral conservation law is applied to the 2-dimensional plane strain problem of two infinite isotropic elastic wedges with opening angles  $\omega'$  and  $\omega''$ , respectively, and with different elastic properties (exhibited through  $E'$ ,  $\nu'$  and  $E''$ ,  $\nu''$ , where  $E$  and  $\nu$  denote Young's modulus and Poisson's ratio), which are bonded together along one edge except for a crack of length  $l$  extending from the apex, and whose corners are subjected to a system of normal and shear point loads, given by  $P'$ ,  $Q'$ ,  $P''$  and  $Q''$ . (See Fig. 1.) An application of the M-integral conservation law yields a relation among the parameters  $P$ ,  $Q$ ,  $E$ ,  $\nu$ ,  $l$  and the energy release rate of the crack. (Symbols without primes refer to both materials.) It is clear that the complexity of the problem precludes the determination of the energy release rate by first solving the corresponding boundary value problem.

The M-integral is given by

$$M = \int_C (W n_i x_i - T_k u_{k,i} x_i) ds \quad (1)$$

where  $W$  is the elastic energy density,  $n_i$  is the unit normal to  $C$  (which we take to be directed to the right when  $C$  is traversed in a given direction) and  $T_k$  is the traction acting on the material to the left of  $C$ . The stresses  $\sigma_{ij} = \sigma_{ji}$  are related to the elastic energy  $W$  and the strains  $\epsilon_{ij} = (u_{i,j} + u_{j,i})/2$  by  $2W = \sigma_{ij} u_{i,j}$ , where  $u_i$  denote the displacements. The traction  $T_k$  is given by  $T_k = \sigma_{ik} n_i$ . The elastic body is assumed to be in equilibrium without body forces, i.e.  $\sigma_{ij,j} = 0$ , and the stress-strain relation is given by  $\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}}$ . The conservation law for  $M$  is that  $M = 0$  whenever  $C$  is a closed path surrounding a simply connected region in the body.

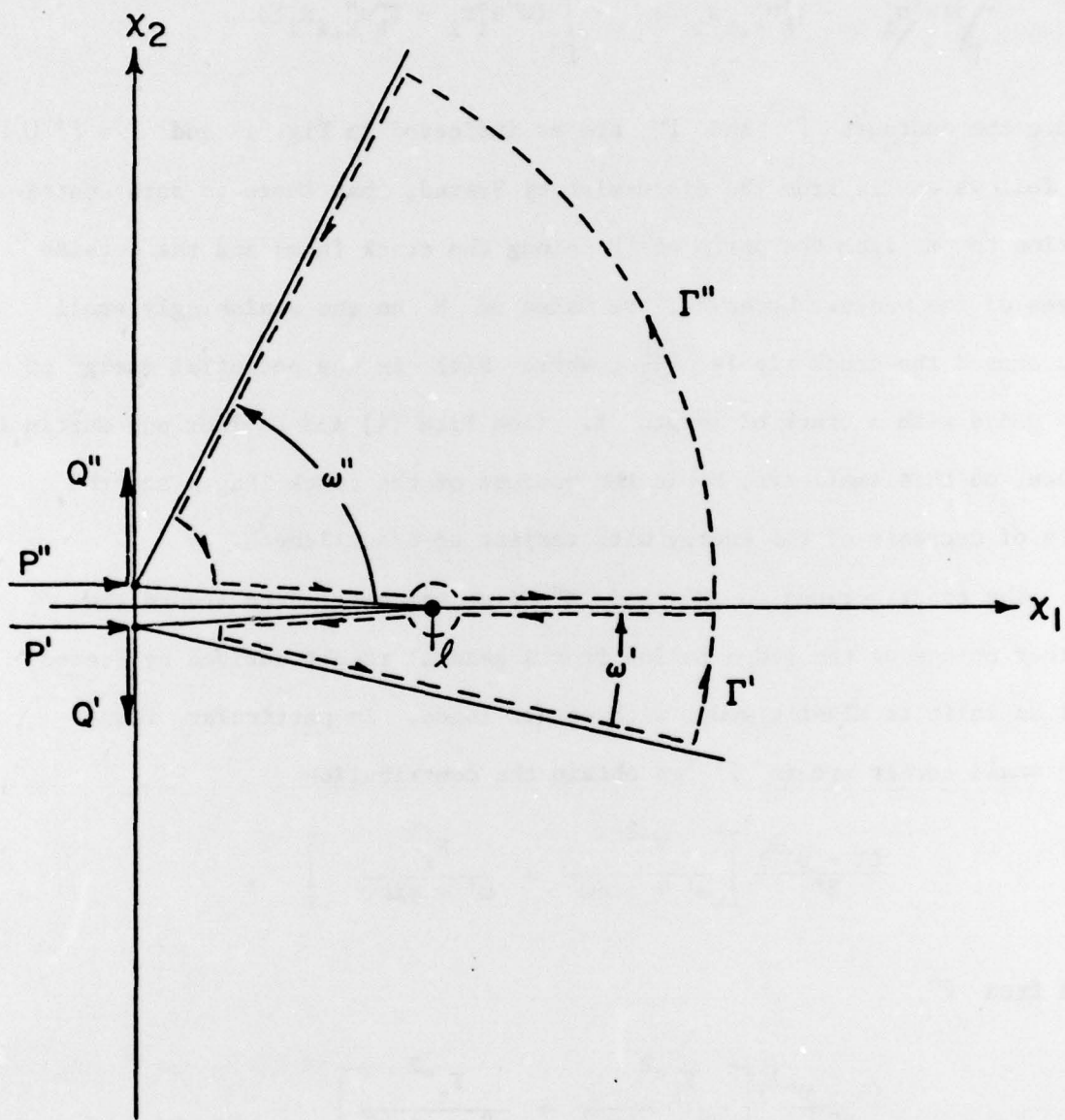


Figure 1.



Consider the integral

$$M = M' + M''$$

$$\begin{aligned} &= \int_{\Gamma} (W n_i x_i - T_k^u u_{k,i} x_i) ds \\ &= \int_{\Gamma'} (W' n'_i x_i - T'_k u'_{k,i} x_i) ds + \int_{\Gamma''} (W'' n''_i x_i - T''_k u''_{k,i} x_i) ds, \end{aligned}$$

where the contours  $\Gamma'$  and  $\Gamma''$  are as indicated in Fig. 1 and  $\Gamma = \Gamma' \cup \Gamma''$ .

It follows easily from the discussion by Freund, that there is zero contribution to  $M$  from the parts of  $\Gamma$  along the crack faces and the outside edges of the wedge. Moreover, the value of  $M$  on the vanishingly small arc around the crack tip is  $\ell \frac{dP}{d\ell}$ , where  $P(\ell)$  is the potential energy of the wedge with a crack of length  $\ell$ . (See Rice [4] and Smelzer and Gurtin [2].) Hence, on this small arc,  $M$  is the product of the crack length and the rate of decrease of the energy with respect to crack length.

The contributions to  $M'$  and  $M''$  from the small arcs around the corner points of the wedge follow from a general result derived by Freund for an infinite elastic wedge with corner loads. In particular, from the small corner arc in  $\Gamma'$  we obtain the contribution

$$\frac{(1 - \nu'^2)}{E'} \left[ \frac{F_a'^2}{\omega' + \sin \omega'} + \frac{F_t'^2}{\omega' - \sin \omega'} \right],$$

and from  $\Gamma''$

$$\frac{(1 - \nu''^2)}{E''} \left[ \frac{F_a''^2}{\omega'' + \sin \omega''} + \frac{F_t''^2}{\omega'' - \sin \omega''} \right],$$

where, employing Freund's notation,

$$F_a' = -P' \cos(\omega'/2) - Q' \sin(\omega'/2),$$

$$F_t' = P' \sin(\omega'/2) - Q' \cos(\omega'/2),$$

$$F_a'' = -P'' \cos(\omega''/2) - Q'' \cos(\omega''/2),$$

$$F_t'' = -P'' \sin(\omega''/2) + Q'' \cos(\omega''/2).$$

On the bond line,  $x_2 = 0$ ,  $x_1 > l$ , the contributions to  $M$  from  $\Gamma'$  and  $\Gamma''$  cancel, since, among the stress and displacement components, only  $\sigma_{11}$  is discontinuous across the interface, and on that line  $n_1 x_1 = 0$  (because the interface is radial) and  $T_{k,i}^u x_i = (u_{1,1} \sigma_{21} + u_{2,1} \sigma_{22}) x_1$ .

It remains to determine the contribution from the large arc for which it suffices to know the far field solution for two wedges bonded together with no crack and with apex loads  $P' + P''$  and  $Q' - Q''$ . From the analysis presented by Bogy [3] for two bonded quarter planes, it is clear that the far field is radial. More precisely, if the elastic fields are represented with respect to polar coordinates  $(r, \theta)$ , then  $r\sigma_{\theta\theta}$  and  $r\sigma_{r\theta}$  vanish for large  $r$  uniformly in  $\theta$ , whereas  $r\sigma_{rr}$  does not. Consequently, on the large arc we may assume  $\sigma_{r\theta}$  and  $\sigma_{\theta\theta}$  are zero. It is easy to see that in this case, to satisfy the equilibrium equations and the compatibility equation, we must take for the stress field

$$\sigma_{rr} = -\frac{A \cos \theta + B \sin \theta}{r}, \quad \sigma_{r\theta} = \sigma_{\theta\theta} = 0. \quad (2)$$

The four constants  $A'$ ,  $B'$ ,  $A''$  and  $B''$  may be calculated from the four conditions

$$u_\theta'(r, \theta \rightarrow 0-) = u_\theta''(r, \theta \rightarrow 0+), \quad (3)$$

$$u_r'(r, \theta \rightarrow 0-) = u_r''(r, \theta \rightarrow 0+), \quad (4)$$

$$\int_{-\omega'}^0 r \sigma'_{rr}(r, \theta) \cos(\theta) d\theta + \int_0^{\omega''} r \sigma''_{rr}(r, \theta) \cos(\theta) d\theta = -(P' + P''), \quad (5)$$

$$\int_{-\omega'}^0 r \sigma'_{rr}(r, \theta) \sin(\theta) d\theta + \int_0^{\omega''} r \sigma''_{rr}(r, \theta) \sin(\theta) d\theta = (Q' - Q''). \quad (6)$$

Equations (3) and (4) assert continuity for the displacements on the bond line, while equations (5) and (6) express the equilibrium of tractions on the wedge  $-\omega' \leq \theta \leq \omega''$ ,  $0 \leq r \leq R$ , where  $R$  is the radius of the large arc. A simple calculation shows that (5) and (6) reduce to

$$A'(1 - \cos 2\omega') - B'(2\omega' - \sin 2\omega') - A''(1 - \cos 2\omega'') - B''(2\omega'' - \sin 2\omega'') = 4(Q' - Q'') \quad (7)$$

$$-A'(2\omega' + \sin 2\omega') + B'(1 - \cos 2\omega') - A''(2\omega'' + \sin 2\omega'') - B''(1 - \cos 2\omega'') = -4(P' + P''). \quad (8)$$

Substitution of (2) into the polar form of the stress-strain law followed by the application of (3) and (4) yields the relations

$$B'/B'' = A'/A'' = (1 + \alpha)/(1 - \alpha), \quad (9)$$

where  $\alpha$  is one of the two Dundurs bi-material parameters given by (See Eogy [3].)

$$\alpha = \begin{cases} \frac{E'(1 - \nu''^2) - E''(1 - \nu'^2)}{E'(1 - \nu''^2) + E''(1 - \nu'^2)} & \text{for plane strain} \\ \frac{E' - E''}{E' + E''} & \text{for generalized plane stress.} \end{cases}$$

It is now an easy matter to solve equations (7), (8) and (9) for  $A'$ ,  $B'$ ,  $A''$  and  $B''$ . In particular, we obtain

$$A' = (a(Q' - Q'') + b(P' + P''))(4/d)$$

$$B' = (-a(P' + P'') - c(Q' - Q''))(4/d)$$



where  $a = (1 - \cos 2\omega') - \left(\frac{1 - \alpha}{1 + \alpha}\right)(1 - \cos 2\omega'')$

$$b = -(2\omega' - \sin 2\omega') - \left(\frac{1 - \alpha}{1 + \alpha}\right)(2\omega'' - \sin 2\omega'')$$

$$c = -(2\omega' + \sin 2\omega') - \left(\frac{1 - \alpha}{1 + \alpha}\right)(2\omega'' + \sin 2\omega'')$$

$$d = a^2 - bc.$$

$A''$  and  $B''$  may now be calculated from (9).

As observed by Freund, for the stress state (2), the integrand of  $M$  is

$$W_{i1} n_i - T_{k,i}^u x_1 = -\frac{1}{2} \frac{(1 - \nu^2)}{E} r \sigma_{rr}^2.$$

Consequently, the contribution to  $M$  from the large arc is

$$\begin{aligned} & -\frac{1}{2} \frac{(1 - \nu'^2)}{E'} \int_{-\omega'}^0 (A' \cos \theta + B' \sin \theta)^2 d\theta - \frac{1}{2} \frac{(1 - \nu''^2)}{E''} \int_0^{\omega''} (A'' \cos \theta + B'' \sin \theta) d\theta \\ & = -\frac{1}{2} \frac{(1 - \nu'^2)}{E'} \left[ \int_{-\omega'}^0 (A' \cos \theta + B' \sin \theta)^2 d\theta + \left(\frac{1 - \alpha}{1 + \alpha}\right) \int_0^{\omega''} (A' \cos \theta + B' \sin \theta) d\theta \right] \quad (10) \end{aligned}$$

$$= -\frac{(1 - \nu'^2)}{E'} (b(P' + P'')^2 + c(Q' - Q'')^2 + 2a(P' + P'')(Q' - Q''))(2/d). \quad (11)$$

Line (10) follows from (9) and the observation that

$\frac{(1 - \nu''^2)}{E''} = \left(\frac{1 + \alpha}{1 - \alpha}\right) \frac{(1 - \nu'^2)}{E'}$ ; whereas, line (11) is derived by simple but tedious algebraic manipulations.



Combining the contributions to  $M$  from the large arc, the two small arcs at the apex and the arc around the crack tip and appealing to the conservation law, we obtain

$$\begin{aligned} \ell \frac{dP}{d\ell} = & \frac{(1 - \nu'^2)}{E'} [-b(P' + P'')^2 + c(Q' - Q'')^2 + 2a(P' + P'')(Q' - Q'')](2/d) \\ & + \frac{1}{2}(P'^2(2\omega' - \sin 2\omega') + Q'^2(2\omega' + \sin 2\omega') + 2P'Q'(\cos 2\omega' - 1))/(\omega'^2 - \sin^2 \omega') \\ & + \frac{1}{2}(\frac{1+\alpha}{1-\alpha})(P''^2(2\omega'' - \sin 2\omega'') + Q''^2(2\omega'' + \sin 2\omega'') + 2P''Q''(\cos 2\omega'' - 1))/(\omega''^2 - \sin^2 \omega'')]. \end{aligned} \quad (12)$$

An important special case of (12) is that of two bonded dissimilar quarter planes with an edge interface crack. Setting  $\omega' = \omega'' = \pi/2$  in (12) yields

$$\begin{aligned} \ell \frac{dP}{d\ell} = & \frac{(1 - \nu'^2)}{E'} [(P' + P'')^2 + (Q' - Q'')^2 - \frac{4\alpha}{\pi} (P' + P'')(Q' - Q'')(1 + \alpha)/((2\alpha)^2 - \pi^2) \\ & + 2((P'^2 + (\frac{1+\alpha}{1-\alpha})P''^2) + (Q'^2 + (\frac{1+\alpha}{1-\alpha})Q''^2) - \frac{4}{\pi} (P'Q' + (\frac{1+\alpha}{1-\alpha})P''Q''))/(\pi^2 - 4)]. \end{aligned} \quad (13)$$

It should be noted that if  $P' = P'' \equiv P$ ,  $Q' = Q'' \equiv Q$  and  $\alpha = 0$ , then

(13) reduces to the result obtained by Freund for identical quarter planes.

Of course, when  $\alpha = 0$ , the response of the bonded quarter planes is the same as for a homogeneous half-space; as described by Dundurs [5], the two quarter planes are "consonant in tension parallel to the interface."

Another interesting case in (12) is when  $\alpha = \pm 1$ . Due to symmetry we consider only  $\alpha = -1$  corresponding to which (12) becomes

$$\epsilon \frac{dP}{d\lambda} = \frac{(1 - \nu'^2)}{2E'} [P'^2(2\omega' - \sin 2\omega') + Q'^2(2\omega' + \sin 2\omega') + 2P'Q'(\cos 2\omega' - 1)] / (\omega'^2 - \sin^2 \omega').$$

It should not be surprising that in (14) only  $P'$  and  $Q'$  appear, since  $\alpha = -1$  corresponds to  $E'' = \infty$ .

It should also be noted that for  $P' = P'' \equiv P$ ,  $Q' = Q'' \equiv Q$  and  $\omega' = \omega'' = \pi/2$ , the level curves of  $\ell \frac{dP}{d\ell}$  in the  $(P, Q)$ -plane are ellipses (straight lines if  $\alpha = 0$ ) centered at  $(0, 0)$ . From this we may conclude as did Freund, that crack extension may result when the unloading of  $P$  and  $Q$  occurs along certain paths in the  $(P, Q)$ -plane. Moreover, this obviously is the case also in (14) with  $\omega' \neq \pi/2$ . It is evident in (12), that in general, regardless of the values of  $\omega'$  and  $\omega''$  and  $\alpha$ , this phenomenon is to be expected.

The authors wish to acknowledge with much appreciation several helpful and encouraging discussions on this problem with Professor L. B. Freund.

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		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) A. Nachman and J. R. Walton		8. CONTRACT OR GRANT NUMBER(s) F49620-79-C-0076 DAAG29-75-C-0024 77-3290
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 3 - Applications of Mathematics
11. CONTROLLING OFFICE NAME AND ADDRESS  See Item 18 below		12. REPORT DATE November 1979
		13. NUMBER OF PAGES 10
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report)  UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report)  Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES U. S. Army Research Office                      Air Force Office of Scientific Research P.O. Box 12211                                      Washington, D. C. 20332 Research Triangle Park North Carolina 27709		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)  Energy release rate, interface edge crack, M-conservation integral		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) A certain conservation integral, the so called M-integral, recently exploited by L. B. Freund in the calculation of stress intensity factors for cracks in homogeneous elastic media, is applied to the calculation of energy release rates for interface edge cracks. Specifically, for an edge crack along the interface between two elastic wedges of different opening angles and dissimilar elastic properties, and that is subjected to point loads at the apex, a relation is derived among the length of the crack, the energy release rate of the crack, the applied loads, the wedge angles and the material parameters.		